

Entropic convergence and the linearized limit for the Boltzmann equation with external force

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ABSTRACT. The purpose of this note, as a compendium, is to extend the results on entropic convergence and the linearized limit for the Boltzmann equation (without external force) in [16] by Levermore to the case of the Boltzmann equation with external force. More specifically, starting from the Boltzmann equation with an external force introduced in [4] by Arsénio and Saint-Raymond, we find conditions on the force, to maintain the result in [16] (as an important application of the theory of DiPerna-Lions renormalized solutions) about the validity of the linearization approximation when the initial datum approaches a global Maxwellian.

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1. INTRODUCTION

Renormalized solutions of the Boltzmann equation (without external forces for simplicity or with self-consistent Vlasov-type forces) are known to exist since the late eighties, thanks to DiPerna and Lions [10]. Subsequently, the convergence of these solutions in the incompressible Navier-Stokes limit was studied by Bardos, Golse and Levermore [5] with some partial success. In that study, a notion of *entropic convergence* was first introduced and was used as a natural tool to obtain strong convergence results for fluctuations about a global Maxwellian. In [16], making use again the notion of entropic convergence as a major tool, Levermore established the convergence of such fluctuations of DiPerna-Lions renormalized solutions to a solution of the linearized Boltzmann equation posed in L^2 space, for the case without external force. It is then natural to think about extending these results to kinetic equations with external forces (thanks to Caglioti) and with convex entropy so that the notion of entropic convergence can be applied to them (thanks to Levermore).

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With this motivation, we thus start with a simple case, the Boltzmann equation with an external regular force F such that

$$\operatorname{div}_v F = 0, \quad F \cdot v = 0 \quad \text{and} \quad F \in L^1_{\text{loc}}(dt; W^{1,1}_{\text{loc}}(Mdvdx)),$$

whose existence of DiPerna-Lions renormalized solutions has been justified by Arsénio and Saint-Raymond in [4] (Section 4.1). Toward studying the strong linearized limit, we require an additional **condition**

$$F \in L^1_{\text{loc}}(dtdx; L^2(Mdv)).$$

In this new setting, we preserve, as in the main Proposition (3.8), the result obtained by Levermore [16] on the linearization approximation: any sequence of DiPerna-Lions renormalized solutions of the Boltzmann equation are shown to have fluctuations (about an equilibrium M) that converge entropically (and hence strongly in L^1) to the solution of the linearized (about M) Boltzmann equation for all positive time, provided that its initial fluctuations (about M) converge entropically to the L^2 initial data of the linearized Boltzmann equation.

Here, the smoothness assumption on the external force F as $F \in L^1_{\text{loc}}(dt; W^{1,1}_{\text{loc}}(Mdvdx))$ can be weakened as long as one can show that with this weakening condition, the DiPerna-Lions renormalized solutions for (2.1) still exist. More generally, we notice that, providing an external force F such that the renormalized solutions exist, the linearization approximation (Proposition 3.8) holds with minimal requirements:

$$\operatorname{div}_v F = 0, \quad F \cdot v = 0, \quad F \in L^1_{\text{loc}}(dtdx; L^2(Mdv)).$$

Regarding application, it is an expectation that given the linearized Boltzmann equation with external force, its fluid dynamical limits [6] can be established in the same spirit of this paper and [16], deriving some kind of equations involving the external force. Also, there is a conjecture that this result on linearization approximation with an external force can be helpful for physics such as for investigation of physical processes in low-temperature weakly-ionized plasmas, for example in the solar photosphere [23, 24, 9].

Section 2 contains preliminary material regarding the Boltzmann equation with external force and its linearization, including a statement about the DiPerna-Lions renormalized solutions thanks to Arsénio and Saint-Raymond [4], in our context. We also make a few crucial preliminary reductions for the proof of our main result (Proposition 3.8). Since the preliminary material introduced contains a subset of that in the first section of [5], the second section of [16] and chapter 1 of part 1 as well as chapter 4 of part 2 of [4], the familiar reader can pass over it lightly.

Section 3 reintroduces the notion of entropic convergence (used in [5, 16, 21, 4]) and contains all the main results. The proofs of some key propositions and theorems have been presented in [5, 16, 4] and so are not reproduced here, but all such propositions and theorems are fully stated for completeness.

Some open problems of extending these results to other settings and other kinetic equations are also discussed.

2. NOTION AND PRELIMINARIES

Being motivated by the work of Levermore in [16], we investigate, in the present paper, the linearized limit of the Boltzmann equation with external force stated by Arsénio and Saint-Raymond in [4] (Section 4.1). This equation is of the form:

$$(2.1) \quad \partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = B(f, f),$$

with a given force field $F(t, x, v)$ satisfying, at least,

$$F, \operatorname{div}_v F \in L^1_{\text{loc}}(dt dx; L^1(M dv)).$$

These conditions on the force field are minimal requirements so that it is possible to define renormalized solutions of (2.1). We will, however, further restrict the range of applicability of force fields:

- we assume $\operatorname{div}_v F = 0$ so that the local conservation of mass is verified;
- we assume $F \cdot v = 0$ so that the global Maxwellian $M(v)$ is an equilibrium state of (2.1). Also, this assumption is crucial because it simultaneously helps simplify the linearized Boltzmann equation.

Here, the mass density $f(t, x, v) \geq 0$, where $t \in [0, \infty)$, $x \in \Omega \subset \mathbb{R}^3$, $v \in \mathbb{R}^3$, represents the distribution of particles which, at time t , are at position x and have velocity v . The Boltzmann collision operator is of the form

$$(2.2) \quad B(f, k) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} (f' k'_* - f k_*) b(v - v_*, \sigma) d\sigma dv_*,$$

where

$$f = f(v), f' = f(v'), k_* = k(v_*), k'_* = k(v'_*),$$

with (v', v'_*) given by the σ -representation of the collision operator (according to the terminology in [26])

$$(2.3) \quad v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma.$$

Another useful representation of B (as in [16]) is of the form

$$(2.4) \quad v' = v - (v - v_*, \sigma) \sigma, \quad v'_* = v_* + (v - v_*, \sigma) \sigma.$$

We refer to [25] for the link between the two representations. In this paper, we make use of the σ -representation. Assuming that the collisions are *elastic*: the momentum and energy for particle pairs during collisions are conserved, i.e.

$$(2.5) \quad v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2,$$

ones can readily show that the quadruple (v, v_*, v', v'_*) parametrized by $\sigma \in \mathbb{S}^2$ provides the family of all solutions to these four equations (2.5). In (2.5), (v, v_*) denote the pre-collisional velocities and (v', v'_*) denote the post-collisional velocities of two interacting particles.

The *collision kernel* (or *cross-section*, by abuse language) measures in some sense the statistical repartition of post-collisional velocities, given the pre-collisional velocities. It depends critically on the nature of the microscopic interactions, and is of the form

$$b(z, \sigma) = b\left(|z|, \frac{z}{|z|} \cdot \sigma\right) \geq 0.$$

The support of b is assumed to be sufficiently large for the Boltzmann H -theorem to hold. Moreover, it will be assumed that b satisfies the bounds

$$(2.6) \quad 0 \leq b(v - v_*, \sigma) \leq C(1 + |v - v_*|^2),$$

for some finite constant C independent of σ . This condition is met by classical Boltzmann collision kernels with a small deflection cut-off [7], meaning $b(z, \sigma) \in L^1_{\text{loc}}(\mathbb{R}^3 \times \mathbb{S}^2)$, i.e. at least locally integrable. It is as a short-range assumption from the physical point of view.

There are well-known facts (see [8, 26]) that the pre-postcollisional changes of variables or simply collisional symmetries

$$(2.7) \quad \begin{aligned} (v, v_*, \sigma) &\rightarrow (v', v'_*, a), \quad \left(\text{with } \sigma = \frac{v' - v'_*}{|v' - v'_*|}, a = \frac{v - v_*}{|v - v_*|}\right), \\ (v, v_*, \sigma) &\rightarrow (v_*, v, -\sigma) \end{aligned}$$

are involutive, and have therefore unit Jacobian determinants. Moreover, as a consequence of microreversibility, they leave the collision kernel b invariant.

Subsequently, for all $f(v), k(v)$ and $\varphi(v)$ regular enough, ones can show that

$$(2.8) \quad \begin{aligned} &\int_{\mathbb{R}^3} B(f, k) \varphi(v) dv \\ &= \frac{1}{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} (f'k'_* - fk_*) b(v - v_*, \sigma) (\varphi + \varphi_* - \varphi' - \varphi'_*) d\sigma dv_* dv. \end{aligned}$$

On the right hand side of (2.8), whenever φ satisfies the functional equation

$$(2.9) \quad \varphi(v) + \varphi(v_*) = \varphi(v') + \varphi(v'_*) \quad \forall (v, v_*, \sigma) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2,$$

then at least formally the integral vanishes. The phrase “at least formally” means that the preceding equations must be rigorously justified with the help of some integrability estimates on the solutions to the Boltzmann equation (2.1). It follows from (2.5) that this integral vanishes if and only if $\varphi(v)$ is a *collision invariant*, i.e. any linear combination of $1, v_1, v_2, v_3, |v|^2$. The proof of this assertion is far from obvious; see for example [7].

Throughout this work, we assume that any solution $f(t, x, v)$ of the Boltzmann equation (2.1) is locally integrable and rapidly decaying in v for each (t, x) .

Thus, taking into account the conditions on the given force field at the beginning of this section, successively multiplying the Boltzmann equation (2.1) by the collision invariants

$1, v_1, v_2, v_3, |v|^2$, then integrating in velocity yields formally the *local conservation laws*:

$$(2.10) \quad \begin{aligned} \partial_t \int_{\mathbb{R}^3} f dv + \operatorname{div}_x \int_{\mathbb{R}^3} v f dv &= 0, \\ \partial_t \int_{\mathbb{R}^3} v f dv + \operatorname{div}_x \int_{\mathbb{R}^3} (v \otimes v) f dv &= \int_{\mathbb{R}^3} F f dv, \\ \partial_t \int_{\mathbb{R}^3} \frac{1}{2} |v|^2 f dv + \operatorname{div}_x \int_{\mathbb{R}^3} \frac{1}{2} |v|^2 v f dv &= 0. \end{aligned}$$

Respectively, they are the local conservation of mass, momentum and energy, which provide the link to a macroscopic description of the gas.

The symmetries of the collision operator B also lead to the other very important feature of the Boltzmann equation. Without caring about integrability issues, we plug $\varphi = \log f$ into the symmetrized integral in (2.8) to derive the *entropy dissipation functional* (or *entropy production functional*, according to the conventions discussed in [26] by Villani):

$$(2.11) \quad \begin{aligned} D(f) &= - \int_{\mathbb{R}^3} B(f, f) \log f dv \\ &= \frac{1}{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} (f' f'_* - f f_*) \log \left(\frac{f' f'_*}{f f_*} \right) b(v - v_*, \sigma) d\sigma dv_* dv \geq 0. \end{aligned}$$

The so-defined *entropy dissipation* (or *entropy production* [26]) $\int_{\mathbb{R}^3} D(f)(t, x) dx$ is non-negative, and the functional

$$\int_0^t \int_{\mathbb{R}^3} D(f)(s, x) dx ds$$

is thus nondecreasing on $t > 0$.

When $B(f, f) = 0$, it is possible to show, since necessarily $D(f) = 0$ in this case, the minimizers of this entropy dissipation functional are the so-called *global Maxwellian distributions* $M_{\rho, u, \theta}$ (constant in space and time) defined by

$$(2.12) \quad M_{\rho, u, \theta}(v) = \frac{\rho}{(2\pi\theta)^{\frac{3}{2}}} e^{-\frac{|v-u|^2}{2\theta}},$$

where $\rho > 0, u \in \mathbb{R}^3, \theta > 0$ are respectively the macroscopic density, bulk velocity and temperature, under some appropriate choice of units. The equation $B(f, f) = 0$ expresses the fact that collisions are no longer responsible for any variation in the density and thus, the gas has reached statistical *equilibrium*.

Let us now introduce the *Boltzmann H's functional* (or *quantity of information* [26]) as

$$(2.13) \quad H(f) := \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(x, v) \log f(x, v) dx dv.$$

The *entropy* associated with f is defined as

$$(2.14) \quad S(f) = -H(f).$$

Then, the non-negativeness of the entropy dissipation leads to the famous *Boltzmann's H theorem*, also known as the second principle of thermodynamics, stating that the H -function is (at least formally) a Lyapunov functional for the Boltzmann equation. Indeed, formally multiplying the Boltzmann equation in (2.1) by $\log f$ and then integrating in space and velocity readily leads to

$$(2.15) \quad \frac{d}{dt} \int_{\mathbb{R}^3} f \log f(t, x, v) dv + \operatorname{div}_x \int_{\mathbb{R}^3} f \log f(t, x, v) v dv + D(f)(t, x) = 0,$$

Equivalently,

$$(2.16) \quad \frac{d}{dt} \int_{\mathbb{R}^3} f \log f(t, x, v) dv = -v \cdot \nabla_x \int_{\mathbb{R}^3} f \log f(t, x, v) dv - D(f)(t, x).$$

For any particle number density $f \geq 0$ and any global Maxwellian distribution $M_{\rho, u, \theta}$, we define the (global) *relative entropy* of f with respect to $M_{\rho, u, \theta}$ by

$$(2.17) \quad H(f|M_{\rho, u, \theta})(t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left(f \log \frac{f}{M_{\rho, u, \theta}} - f + M_{\rho, u, \theta} \right) (t) dx dv \geq 0,$$

which provides a natural proximity of f to that global equilibrium $M_{\rho, u, \theta}$. This choice of H as the entropy functional (2.17) is based on the fact that its integrand is a non-negative strictly convex function of f with a minimum value of 0 at $f = M$. For simplicity, we denote the relative entropy by $H(f)$, whenever the relative Maxwellian distribution is clearly implied.

Throughout this work, we will restrict our attention to the case of a spatial domain without boundary, in particular, a periodic box \mathbb{T}^3 (as a torus), thus avoiding all technical obstacles due to boundaries. Assuming here that the gas is at equilibrium at infinity, without loss of generality, we consider linearizations about the dimensionless Maxwellian

$$(2.18) \quad M(v) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{|v|^2}{2}}.$$

The condition $F \cdot v = 0$ then ensures that the global Maxwellian $M(v)$ is an equilibrium state of (2.1). Indeed, plugging $f = M(v)$ into (2.1), we obtain

$$B(M, M) = \partial_t M + v \cdot \nabla_x M + F \cdot \nabla_v M = F \cdot (-vM) = (-M)(F \cdot v) = 0.$$

Finally, the units of length are chosen so that \mathbb{T}^3 has a unit volume, leading to the normalizations

$$(2.19) \quad \int_{\mathbb{S}^2} d\sigma = 1, \quad \int_{\mathbb{R}^3} M dv = 1, \quad \int_{\mathbb{T}^3} dx = 1,$$

Since $M dv$ is a positive unit measure on \mathbb{R}^3 , we denote by $\langle \xi \rangle$ the average over this measure of any integrable function $\xi = \xi(v)$,

$$(2.20) \quad \langle \xi \rangle = \int_{\mathbb{R}^3} \xi M dv.$$

From now on, we are interested in the fluctuations of a density $f(t, x, v)$ around $M(v)$ defined in (2.18), as a global normalized Maxwellian. It is thus natural to employ the relative density $G(t, x, v)$ defined by $F = MG$. Recasting the Boltzmann equation (2.1) for G yields

$$(2.21) \quad \begin{aligned} \partial_t G + v \cdot \nabla_x G + F \cdot \nabla_v G &= Q(G, G), \\ G(0, x, v) &= G^{\text{in}}(x, v) \geq 0, \end{aligned}$$

where we denote

$$(2.22) \quad Q(G, K) = \frac{1}{M} B(MG, MK).$$

Hence,

$$(2.23) \quad Q(G, G) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} (G'G'_* - GG_*) b(v - v_*, \sigma) d\sigma M_* dv_*.$$

The bound (2.6) on the collision kernel b ensures that

$$(2.24) \quad \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(v - v_*, \sigma) d\sigma M_* dv_* M dv < \infty.$$

Since $d\mu \equiv b(v - v_*, \sigma) d\sigma M_* dv_* M dv$ is a non-negative finite measure on $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2$, we denote by $\langle\langle \Phi \rangle\rangle$ the integral over this measure of any integrable function $\Phi = \Phi(v, v_*, \sigma)$,

$$(2.25) \quad \langle\langle \Phi \rangle\rangle = \int \Phi d\mu.$$

If G solves the Boltzmann equation (2.21), then it satisfies the local entropy dissipation law as in (2.15)

$$(2.26) \quad \begin{aligned} \partial_t \langle G \log G - G + 1 \rangle + \text{div}_x \langle v (G \log G - G + 1) \rangle \\ = -\frac{1}{4} \langle\langle \log \left(\frac{G'G'_*}{GG_*} \right) (G'G'_* - GG_*) \rangle\rangle \leq 0. \end{aligned}$$

Furthermore, integrating this over space and time gives the *global entropy equality*

$$(2.27) \quad H(G(t)) + \int_0^t R(G(s)) ds = H(G^{\text{in}}),$$

where $H(G)$ is the entropy functional

$$(2.28) \quad H(G) = \int_{\mathbb{T}^3} \langle G \log G - G + 1 \rangle dx,$$

and $R(G)$ is the entropy dissipation rate functional

$$(2.29) \quad R(G) = \int_{\mathbb{T}^3} D(G) dx = \int_{\mathbb{T}^3} \frac{1}{4} \langle\langle (G'G'_* - GG_*) \log \left(\frac{G'G'_*}{GG_*} \right) \rangle\rangle dx.$$

From (2.17), notice that

$$(2.30) \quad H(G) = H(G|1) \geq 0, \quad \text{and} \quad H(G) = 0 \quad \text{iff} \quad G = 1.$$

Recall that the prefixes w - or w^* - express that a given space is endowed with its weak or weak-* topology, respectively. About the notation regarding spaces, see Appendix A.

2.1. Renormalized solutions of the Boltzmann equation. DiPerna and Lions [10] proved the existence of a temporally global weak solution to the Boltzmann equation without external force (i.e. the case $F = 0$ in (2.1), (2.21)) over the spatial domain \mathbb{R}^3 for any initial data satisfying natural physical bounds. With slight modifications, their theory can be extended to the periodic box \mathbb{T}^3 . In the recent work [4], Arsénio and Saint-Raymond have justified this existence of a global weak solution to the Boltzmann equation with external force (2.21), for large initial data $G(0, x, v) = G^{\text{in}}(x, v)$. Their construction based on the notion of renormalized solutions originated from [10]. Toward investigating the linearized limit, we state again these results from [4] in our context, with the benefit of the notations and results from [16].

Definition 2.1 ([10, 12, 5, 16, 4]). *We say that a nonlinearity $\beta \in C([0, \infty); \mathbb{R})$ is an admissible **renormalization** and correspondingly $N \in C([0, \infty); (0, \infty))$ is an admissible **normalization** if they satisfy, for some finite constant $C > 0$,*

$$|\beta'(z)| = \frac{1}{N(z)} \leq \frac{C}{1+z} \leq \frac{C}{(1+z)^{\frac{1}{2}}} \quad \forall z \geq 0,$$

where

$$\beta'(z) = \frac{1}{N(z)}.$$

A relative density functions $G(t, x, v) \geq 0$, where $(t, x, v) \in [0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3$, such that

$$(2.31) \quad G \in C([0, \infty); w\text{-}L^1_{\text{loc}}(M dv dx)) \cap L^\infty([0, \infty), dt; L^1_{\text{loc}}(dx; L^1((1 + |v|^2) M dv))),$$

is a **renormalized solution of the Boltzmann equation** (2.21) if it solves

$$(2.32) \quad \begin{aligned} \partial_t \beta(G) + v \cdot \nabla_x \beta(G) + F \cdot \nabla_v \beta(G) &= \frac{1}{N(G)} Q(G, G), \\ G(0, x, v) &= G^{\text{in}}(x, v), \end{aligned}$$

in the sense of distributions for any admissible renormalization (correspondingly, normalization), and satisfies the entropy inequality, for all $t \geq 0$,

$$(2.33) \quad H(f(t)) + \int_0^t \int_{\mathbb{T}^3} D(f(s)) dx ds \leq H(f^{\text{in}}) < \infty,$$

where $f^{\text{in}} = M G^{\text{in}}$ is initial value of $f = M G$ and the relative entropy $H(f) = H(f|M)$ is defined in (2.17), while the entropy dissipation functional D is defined in (2.11). In this definition, equivalently, $f = M G$ is called a renormalized solution of the Boltzmann equation (2.1).

Notice from [4] that the renormalized collision $\beta'(G)Q(G, G)$ is well-defined in

$$L^1_{\text{loc}}(dtdx; L^1(Mdv))$$

for any admissible renormalization (correspondingly, normalization), any function G in (2.31) and any integrable collision kernel $b(z, \sigma) \in L^1_{\text{loc}}(\mathbb{R}^3 \times \mathbb{S}^2)$ satisfying the so-called DiPerna-Lions assumption

$$(2.34) \quad \lim_{|v| \rightarrow \infty} \frac{1}{|v|^2} \int_{K \times \mathbb{S}^2} b(v - v_*) dv_* d\sigma = 0,$$

for any compact set subset $K \subset \mathbb{R}^3$.

Indeed, we first consider non-negative renormalizations β satisfying

$$0 \leq \beta'(z) \leq \frac{C}{1+z}.$$

The collision operator $Q(G, G)$ can be decomposed into gain and loss parts, respectively

$$(2.35) \quad \begin{aligned} Q^+(G, G) &= \int_{\mathbb{R}^3 \times \mathbb{S}^2} G' G'_* b(v - v_*, \sigma) d\sigma M_* dv_*, \\ Q^-(G, G) &= \int_{\mathbb{R}^3 \times \mathbb{S}^2} G G_* b(v - v_*, \sigma) d\sigma M_* dv_*. \end{aligned}$$

From the hypothesis (2.34), it is possible to show directly that (see [2], for instance, for more details)

$$\lim_{|v| \rightarrow \infty} \frac{1}{|v|^2} \int_{\mathbb{R}^3 \times \mathbb{S}^2} b(v - v_*) M_* dv_* d\sigma = 0.$$

Thus, the renormalized loss part $\beta'Q^-(G, G)$ is readily estimated as

$$(2.36) \quad \begin{aligned} &\int_{\mathbb{R}^3} \beta'(G) Q^-(G, G) M dv \\ &= \int_{\mathbb{R}^3} \left(\frac{1}{1 + |v_*|^2} \int_{\mathbb{R}^3 \times \mathbb{S}^2} \beta'(G) G b(v - v_*, \sigma) d\sigma M dv \right) (G_*) (1 + |v_*|^2) M_* dv_* \\ &\leq C \|G\|_{L^1((1+|v|^2)Mdv)}. \end{aligned}$$

The renormalized gain part $\beta'Q^+(G, G)$ is well-defined in $L^1_{\text{loc}}(dtdx; L^1(Mdv))$ by the renormalized Boltzmann equation (2.32) since it is the only un-estimated term remaining and it is non-negative. Also in [4], an alternative method based on entropy dissipation control, which was originally performed in [10], is also discussed to claim that the renormalized gain part $\beta'Q^+(G, G)$ is well-defined in $L^1_{\text{loc}}(dtdx; L^1(Mdv))$. These controls are easily extended to signed renormalizations satisfying

$$|\beta'(z)| \leq \frac{C}{1+z}$$

as the Boltzmann equation (2.32) is linear with respect to renormalizations (correspondingly, normalizations) so that $\beta'(z)$ can be decomposed into positive and negative parts.

Finally, it is possible to extend the definition of the renormalized collision operator

$$\beta'(G)Q(G, G)$$

to all admissible renormalizations [4].

Therefore, saying G is a *weak solution* of the renormalized equation (2.32) means that it is initially equal to G^{in} and it should satisfy (as in [16, 5]), for any non-negative test function $\varphi \in L^\infty(Mdv; C^1(\mathbb{T}^3))$ and every $0 \leq t_1 < t_2 < \infty$, that

$$(2.37) \quad \begin{aligned} \int_{\mathbb{T}^3} \langle G(t_2) \rangle \varphi dx - \int_{\mathbb{T}^3} \langle G(t_1) \rangle \varphi dx - \int_{t_1}^{t_2} \int_{\mathbb{T}^3} \langle \beta(G)(v \cdot \nabla_x + F \cdot \nabla_v) \varphi \rangle dx dt \\ = \int_{t_1}^{t_2} \int_{\mathbb{T}^3} \langle \frac{1}{N} Q(G, G) \varphi \rangle dx dt. \end{aligned}$$

Being derived from [4], the following theorem is a modern formulation of the existence result found in [10].

Theorem 2.2 ([4] **Theorem 4.1**, [10, 12]). *Let $b(z, \sigma)$ be a locally integrable collision kernel satisfying the DiPerna-Lions assumption (2.34), and let F be a given force field*

$$F(t, x, v) \in L^1_{\text{loc}}(dt dx; L^1(Mdv))$$

such that

$$(2.38) \quad \text{div}_v F = 0, \quad F \cdot v = 0 \quad \text{and} \quad F \in L^1_{\text{loc}}(dt; W^{1,1}_{\text{loc}}(Mdv dx)).$$

Then, for any initial condition $G^{\text{in}} \in L^1_{\text{loc}}(dx; L^1((1 + |v|^2)Mdv))$ such that $G^{\text{in}} \geq 0$ and

$$H(G^{\text{in}}) = H(G^{\text{in}}|1) = \int_{\mathbb{R}^3 \times \mathbb{T}^3} (G^{\text{in}} \log G^{\text{in}} - G^{\text{in}} + 1) dx Mdv < \infty,$$

there exists a renormalized solution $G(t, x, v)$ to the Boltzmann equation (2.21). Moreover, it satisfies the local conservation of mass

$$(2.39) \quad \partial_t \langle G \rangle + \text{div}_x \langle vG \rangle = 0,$$

and the global entropy inequality, for any time $t \geq 0$,

$$(2.40) \quad H(G(t)) + \int_0^t R(G) ds \leq H(G^{\text{in}}).$$

Here, the relative entropy $H(G) = H(G|1)$ is defined in (2.28), while the entropy dissipation rate is defined in (2.29).

Remark 2.3. *The DiPerna-Lions theory, however, does not assert the local conservation of momentum, the global conservation of energy, or the global entropy equality (2.27); nor does it assert the uniqueness of the solution.*

In the case of long-range interactions (i.e. when the collision kernel is non-integrable), there exist renormalized solutions with a defect measure in the spirit of the construction by Alexandre and Villani [1], which has been addressed by Arsénio and Saint-Raymond [3] for the Vlasov-Maxwell-Boltzmann system.

The proof of this theorem follows the usual steps found in the analysis of weak solutions of partial differential equations based on the work of DiPerna and Lions [10, 12], and of Lions [17, 18, 19]. These steps are often reduce to the study of the crucial weak stability of solutions, as in our case. Also, recently, the proof of this weak stability result has been justified in [4] by studying the weak stability of renormalized solutions of the Boltzmann equation (2.21), equivalently, the weak stability of weak solutions of the renormalized equation (2.32). We summarize here the essential steps of the proof of this Theorem 2.2 based on [4, 10, 12, 17, 18, 19].

Step 1: Let us consider a sequence $\{G_j\}_{j \in \mathbb{N}}$ of actual renormalized solutions to (2.21), with initial data $\{G_j^{\text{in}}\}_{j \in \mathbb{N}}$, which converges weakly (at least in L^1_{loc} , for instance) when $j \rightarrow \infty$ to G^{in} . Furthermore, we assume that the initial data satisfies the following strong entropic convergence

$$\lim_{j \rightarrow \infty} H(G_j^{\text{in}}) = H(G^{\text{in}}),$$

so that the entropy inequality is uniformly satisfied

$$(2.41) \quad H(G_j(t)) + \int_0^t R(G_j(s)) ds \leq H(G_j^{\text{in}}).$$

Thus, thanks to the uniform bound on the entropies $H(G_j(t))$, using Young inequality (see [4]) and the Dunford-Pettis compactness criterion (see [20]), extracting a subsequence if necessary, we may assume that the sequence $\{G_j\}_{j \in \mathbb{N}}$ converges weakly, as $j \rightarrow \infty$, to some G in $L^1_{\text{loc}}(dtdx; L^1(Mdv))$.

Ones now use the idea of renormalized solutions to obtain L^1 bounds on normalized collision operators

$$\frac{Q^\pm(G_j, G_j)}{1 + \delta M G_j *_v (\int_{\mathbb{S}^2} b(\cdot, \sigma) d\sigma)} \quad \text{and} \quad \beta'(G_j) Q^\pm(G_j, G_j),$$

for any $\delta > 0$ and any admissible nonlinearity $\beta(z) \in C^1([0, \infty); \mathbb{R})$. These operators even have “weakly compact in L^1 ” estimates using entropy dissipation (or $(- \text{entropy})$ dissipation...).

At this stage, using the convexity methods from [12], by assuming $\text{div}_v F = 0$ and $F \cdot v = 0$, passing to the limit in (2.41), ones obtain the entropy inequality (2.40), i.e. it follows from formal estimates on the Boltzmann equation (2.21), even when $F \neq 0$.

Step 2: These estimates are shown to prove that various *velocity averages* of $G_j, Q^\pm(G_j, G_j)$ are relatively compact in $L^1(dtdx)$ or a.e., and converge to the averages of $G, Q^\pm(G, G)$. This step relies on a standard use of the velocity averaging theory for linear transport equation (as treated in [13], for instance, with $F \cdot \nabla_v \beta(G_j) = \text{div}_v(F \beta(G_j))$ as a source term).

Step 3: Using the information from the previous steps to pass to the limit by a rather delicate procedure involving the interpretation of the Boltzmann equation (2.21) along almost particle paths, and thus recovering the original equation.

In this step, notice that the regularity hypothesis on F that $F \in L^1_{\text{loc}}(dt; W^{1,1}_{\text{loc}}(Mdvdx))$ is included in the hypotheses of Theorem 2.2 so that the vector field $(v, F(t, x, v)) \in \mathbb{R}^6$ satisfies the conditions imposed on the transport equation from the lemma in [11](Theorem II.1, p. 516). \square

2.2. Linearization of the Boltzmann equation. Linearizing the Boltzmann equation (2.21) about 1, i.e. $G = 1 + g$, yields

$$(2.42) \quad \begin{aligned} \partial_t g + v \cdot \nabla_x g + F \cdot \nabla_v g + Lg &= 0, \\ g(0, x, v) &= g^{\text{in}}(x, v), \end{aligned}$$

where the linearized collision operator, thanks to the defined Q in (2.22), is given by

$$(2.43) \quad Lg \equiv -2Q(1, g) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} (g + g_* - g' - g'_*) b(v - v_*, \sigma) d\sigma M_* dv_*.$$

Notice that, for M as defined in (2.18), this linearization is equivalent to the linearization of the Boltzmann equation (2.1) about M . Indeed, since $f = MG = M(1 + g) = M + Mg$, the Boltzmann equation (2.1) can be written as

$$\partial_t(M + Mg) + v \cdot \nabla_x(M + Mg) + F \cdot \nabla_v(M + Mg) = B(M + Mg, M + Mg).$$

Equivalently,

$$M\partial_t g + Mv \cdot \nabla_x g + F \cdot (-Mv - Mvg + M\nabla_v g) = 2B(M, Mg) + B(Mg, Mg),$$

Taking into account the condition $F \cdot v = 0$ and the defined collision operator Q in (2.22), we obtain

$$\partial_t g + v \cdot \nabla_x g + F \cdot \nabla_v g = 2Q(1, g) + \frac{1}{M}B(Mg, Mg).$$

Therefore, linearizing the Boltzmann equation (2.1) about M also yields the linearized equation of the form (2.42).

Now, the existence and uniqueness of solution of the linearized Boltzmann equation (2.42) are asserted, thanks to the semigroup method discussed in [22] by Scharf. Indeed, the operator L has a non-negative definite self-adjoint Friedrichs extension to the Hilbert space $L^2(Mdv)$ with the inner product $\langle g, k \rangle$. Notice that

$$\langle (v \cdot \nabla_x)g, k \rangle = \int_{\mathbb{T}^3} (kv) \cdot (\nabla_x g) dx = - \int_{\mathbb{T}^3} (gv) \cdot (\nabla_x k) dx = - \langle g, (v \cdot \nabla_x)k \rangle,$$

and

$$\langle (F \cdot \nabla_v)g, k \rangle = \int_{\mathbb{R}^3} (kF) \cdot (\nabla_v g) dv = - \int_{\mathbb{R}^3} (gF) \cdot (\nabla_v k) dv = - \langle g, (F \cdot \nabla_v)k \rangle.$$

Thus, the operators $v \cdot \nabla_x$ and $F \cdot \nabla_v$ are skew-adjoint on $L^2(Mdvdx)$, and

$$-v \cdot \nabla_x - F \cdot \nabla_v - L$$

is a closable dissipative operator on $L^2(Mdvdx)$ that generates a strongly continuous contraction semigroup. Hence, for every $g^{\text{in}} \in L^2(Mdvdx)$, there exists a unique

$$g \in C([0, \infty); L^2(Mdvdx))$$

that solves the linearized Boltzmann equation (2.42). Every such solution g satisfies the weak formulation

$$(2.44) \quad \int_{\mathbb{T}^3} \langle g(t_2) \varphi \rangle dx - \int_{\mathbb{T}^3} \langle g(t_1) \varphi \rangle dx - \int_{t_1}^{t_2} \int_{\mathbb{T}^3} \langle g(v \cdot \nabla_x + F \cdot \nabla_v) \varphi \rangle dx dt + \int_{t_1}^{t_2} \int_{\mathbb{T}^3} \langle (Lg) \varphi \rangle dx dt = 0,$$

for every $\varphi \in L^2(Mdv; C^1(\mathbb{T}^3))$. The solution g also satisfies the dissipation equality (after multiplying both side of the linearized equation (2.42) with g and integrating over time, velocity and space)

$$(2.45) \quad \int_{\mathbb{T}^3} \frac{1}{2} \langle g^2(t) \rangle dx + \int_0^t \int_{\mathbb{T}^3} \frac{1}{4} \langle \langle q^2 \rangle \rangle dx ds = \int_{\mathbb{T}^3} \frac{1}{2} \langle g^{\text{in}2} \rangle dx,$$

where $q = (g' + g'_* - g - g_*)$ in the classical identity $\langle g, Lg \rangle = \frac{1}{4} \langle \langle q^2 \rangle \rangle$. This identity can be obtained by using formulas (2.43), (2.22) and (2.8).

3. THE LINEARIZED LIMIT

This section of the case with external force extends the work (without external force) [16] of Levermore.

Let G_ϵ be a family of DiPerna-Lions renormalized solutions to the scaled Boltzmann initial-value problem (2.21) such that the initial data G_ϵ^{in} satisfies the entropy bound

$$(3.1) \quad H(G_\epsilon^{\text{in}}) \leq C^{\text{in}} \epsilon^2,$$

for some fixed $C^{\text{in}} > 0$. Consider the sequence of fluctuations g_ϵ defined by the relation

$$(3.2) \quad G_\epsilon = 1 + \epsilon g_\epsilon.$$

The DiPerna-Lions entropy inequality (2.40) and the entropy bound (3.1) are consistent with this order of fluctuation about the equilibrium $G = 1$. More specifically, in [16], it is shown that the so-defined sequence g_ϵ is of order one.

With this in mind, the DiPerna-Lions normalization here is chosen to be in the form

$$(3.3) \quad N_\epsilon = N(G_\epsilon) = \frac{2}{3} + \frac{1}{3} G_\epsilon = 1 + \frac{1}{3} \epsilon g_\epsilon.$$

One reason to choose this form is that formally, $N_\epsilon \rightarrow 1$ as $\epsilon \rightarrow 0$; thus, the normalizing factor will conveniently disappear from all algebraic expression considered in this limit. Another reason is that ones wish to simplify the specific encounters during some subsequence estimates. Of course, the main results are independent of this choice of normalization.

Given this choice and let

$$(3.4) \quad \gamma_\epsilon = \frac{1}{\epsilon} \beta(G_\epsilon) = \frac{3}{\epsilon} \left(1 + \frac{1}{3} \epsilon g_\epsilon \right).$$

The renormalized Boltzmann equation (2.32) now becomes

$$(3.5) \quad \partial_t \gamma_\epsilon + (v \cdot \nabla_x + F \cdot \nabla_v) \gamma_\epsilon = \frac{1}{\epsilon} \frac{Q(G_\epsilon, G_\epsilon)}{N_\epsilon}.$$

Since γ_ϵ formally behaves like g_ϵ for small ϵ , it should be thought of as the normalized form of the fluctuations g_ϵ .

The first objective is to characterize properties of the limit of the fluctuations g_ϵ . The priori estimates needed are obtained by the combination of the entropy inequality (2.40) and the entropy bound (3.1) on the initial data:

$$(3.6) \quad H(G_\epsilon(t)) + \int_0^t R(G_\epsilon(s)) ds \leq H(G_\epsilon^{\text{in}}) \leq C^{\text{in}} \epsilon^2.$$

Notice here that the terms involving the dissipation rate R , defined in (2.29), can be thought to formally measure the proximity to equilibrium of the scaled collision integrand

$$(3.7) \quad q_\epsilon = \frac{1}{\epsilon} (G'_\epsilon G'_{\epsilon*} - G_\epsilon G_{\epsilon*}).$$

As in [5], the relative entropy and dissipation rate can be written as

$$(3.8) \quad H(G_\epsilon) = \int_{\mathbb{T}^3} \langle h(\epsilon g_\epsilon) \rangle dx, \quad R(G_\epsilon) = \int_{\mathbb{T}^3} \frac{1}{4} \langle \left\langle r \left(\frac{\epsilon q_\epsilon}{G_{\epsilon*} G_\epsilon} \right) G_{\epsilon*} G_\epsilon \right\rangle \rangle dx,$$

where h, r are convex functions

$$(3.9) \quad h(z) = (1+z) \log(1+z) - z, \quad r(z) = z \log(1+z).$$

Since $h(z) = O(z^2)$ and $r(z) = O(z^2)$ as $z \rightarrow 0$, one easily observe that $H(G_\epsilon)$ and $R(G_\epsilon)$ asymptotically behave almost like L^2 norms of g_ϵ and q_ϵ respectively as $\epsilon \rightarrow 0$. Therefore, based on the entropy and entropy dissipation bounds (resulted from the entropy inequality and bound (3.6)), the following proposition about the weak compactness statements regarding g_ϵ and q_ϵ holds. It follows from Proposition 1 of [16] (derived from Proposition 3.1 and 3.4 of [5]).

Proposition 3.1 ([16], **The infinitesimal fluctuations lemma**). *Consider a sequence*

$$G_\epsilon \in C([0, \infty); w\text{-}L^1_{\text{loc}}(M dv dx)) \cap L^\infty([0, \infty), dt; L^1_{\text{loc}}(dx; L^1((1+|v|^2) M dv)))$$

that satisfies the entropy inequality and bound (3.6), where $G_\epsilon^{\text{in}} = G_\epsilon(0)$. Let g_ϵ and g_ϵ^{in} be the corresponding fluctuations (3.2), and q_ϵ the scaled collision integrands (3.7). Then

- a) the sequence $(1+|v|^2)g_\epsilon(t)$ is relatively compact in $w\text{-}L^1_{\text{loc}}(M dv dx) \quad \forall t \geq 0$;*
- b) any convergence subsequence of $g_\epsilon(t)$ as $\epsilon \rightarrow 0$ has a limit $g(t) \in L^2(M dv dx)$;*
- c) the sequence $(1+|v|^2)g_\epsilon$ is bounded in $L^\infty(dt; L^1(M dv dx))$ and relatively compact in $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(M dv dx))$;*

- d) any convergent subsequence of g_ϵ as $\epsilon \rightarrow 0$ has a limit $g \in L^\infty(dt; L^2(Mdvdx))$;
e) the sequence $(1 + |v|^2)q_\epsilon/N_\epsilon$ is relatively compact in $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(d\mu dx))$;
f) any convergence subsequence of q_ϵ/N_ϵ as $\epsilon \rightarrow 0$ has a limit $q \in L^2(dt; L^2(d\mu dx))$;
g) for almost every $t \geq 0$, any such g and q satisfy the inequalities

$$(3.10) \quad \int_{\mathbb{T}^3} \frac{1}{2} \langle g^2(t) \rangle dx \leq \liminf_{\epsilon \rightarrow 0} \int_{\mathbb{T}^3} \langle \frac{1}{\epsilon^2} h(\epsilon g_\epsilon(t)) \rangle dx,$$

$$(3.11) \quad \int_0^t \int_{\mathbb{T}^3} \frac{1}{4} \langle \langle q^2 \rangle \rangle dx ds \leq \liminf_{\epsilon \rightarrow 0} \int_0^t \int_{\mathbb{T}^3} \frac{1}{4} \langle \langle \frac{1}{\epsilon^2} r \left(\frac{\epsilon q_\epsilon}{G_\epsilon G_{\epsilon*}} \right) G_\epsilon G_{\epsilon*} \rangle \rangle dx ds.$$

$$(3.12) \quad \int_{\mathbb{T}^3} \frac{1}{2} \langle g^2(t) \rangle dx + \int_0^t \int_{\mathbb{T}^3} \frac{1}{4} \langle \langle q^2 \rangle \rangle dx ds \leq \liminf_{\epsilon \rightarrow 0} \int_{\mathbb{T}^3} \langle \frac{1}{\epsilon^2} h(\epsilon g_\epsilon^{\text{in}}) \rangle dx \leq C^{\text{in}}.$$

The proposition above does not involve the fact that the sequence g_ϵ will eventually represent fluctuations of the number density in the Boltzmann equation; and the proof is mainly based on the convexity of the integrands in the entropy inequality (2.40) and the entropy bound (3.1).

We now recall the notion of “*entropic convergence*” introduced first in [5].

Definition 3.2. A family of fluctuations g_ϵ is said to **converge entropically** of order ϵ to $g \in L^2(Mdvdx)$ if and only if

$$(3.13) \quad g_\epsilon \rightharpoonup g \text{ in } w\text{-}L^1(Mdvdx),$$

$$(3.14) \quad \int_{\mathbb{T}^3} \langle \frac{1}{\epsilon^2} h(\epsilon g_\epsilon) \rangle dx \rightarrow \int_{\mathbb{T}^3} \frac{1}{2} \langle g^2 \rangle dx,$$

as $\epsilon \rightarrow 0$. The condition (3.14) is equivalent to

$$(3.15) \quad \frac{1}{\epsilon^2} H(G_\epsilon) \rightarrow \int_{\mathbb{T}^3} \frac{1}{2} \langle g^2 \rangle dx.$$

Throughout this paper, all entropic convergence will be understood to be “of order ϵ ”. That this notion of convergence is stronger than that of the strong $L^1((1 + |v|^2)Mdvdx)$ topology was shown in Proposition 4.11 of [5]. Applying this notion to Proposition 1, the following sharpening of inequality (3.12) arrives.

Proposition 3.3 ([16], **The dissipation inequality corollary**). Consider a sequence

$$G_\epsilon \in C([0, \infty); w\text{-}L^1(Mdvdx))$$

that satisfies the entropy inequality and bound (3.6), where $G_\epsilon^{\text{in}} = G_\epsilon(0)$ has fluctuations $g_\epsilon^{\text{in}} = g_\epsilon(0)$ that converge entropically to some $g^{\text{in}} \in L^2(Mdvdx)$. Let g_ϵ and q_ϵ be the corresponding fluctuations (3.2) and scaled collision integrands (3.7), and g and q be the corresponding weak limits. Then

$$(3.16) \quad \int_{\mathbb{T}^3} \frac{1}{2} \langle g^2(t) \rangle dx + \int_0^t \int_{\mathbb{T}^3} \frac{1}{4} \langle \langle q^2 \rangle \rangle dx ds \leq \int_{\mathbb{T}^3} \frac{1}{2} \langle g^{\text{in}2} \rangle dx.$$

Remark 3.4. Thanks to (3.15), the assumption that the initial fluctuations g_ϵ^{in} converge entropically to some g^{in} in $L^2(Mdvdx)$ implies that these fluctuations satisfy the entropy bound (3.1).

For removing the normalization and linearizing the collision integrand in the limit, the technical estimates needed again derive from the entropy inequality and bound (3.6). In contrast with the previous propositions which depend mainly on basic properties of convexity, the estimates in the following proposition rely heavily on the specific form of the entropy functional (2.28). This proposition is contained within Corollary 3.2 and Proposition 3.3 in [5].

Proposition 3.5 ([16], **The bounds of nonlinear terms lemma**). *Consider a sequence*

$$G_\epsilon \in C([0, \infty); w-L^1(Mdvdx))$$

that satisfies the entropy inequality and bound (3.6). Let g_ϵ be the corresponding fluctuations (3.2). Then

- a) if the sequence g_ϵ converges to g in $w-L^1_{\text{loc}}(dt; w-L^1(Mdvdx))$, then the sequence γ_ϵ converges to g in $w-L^1_{\text{loc}}(dt; w-L^1((1 + |v|^2)Mdvdx))$;*
- b) the sequence g_ϵ^2/N_ϵ is bounded in $L^\infty(dt; L^1(Mdvdx))$; and the bound is given by*

$$(3.17) \quad \int_{\mathbb{T}^3} \langle \frac{g_\epsilon^2}{N_\epsilon} \rangle(t) dx \leq 2C^{\text{in}} \quad \forall t \geq 0.$$

- c) as $\epsilon \rightarrow 0$,*

$$(3.18) \quad |v|^2 \frac{g_\epsilon^2}{N_\epsilon} = O\left(\log\left(\frac{1}{\epsilon|\log(\epsilon)|}\right)\right) \quad \text{in } L^\infty(dt; L^1(Mdvdx)).$$

The statement *c* above plays a key role in the following proposition, to control the quadratic terms of the scaled Boltzmann collision integrands in order to derive their linearized limit.

Proposition 3.6 ([16], **The limiting collision integrand theorem**). *Consider a family*

$$G_\epsilon \in C([0, \infty); w-L^1(Mdvdx))$$

that satisfies the entropy inequality and bound (3.6). Let g_ϵ and q_ϵ be the corresponding fluctuations (3.2) and scaled collision integrands (3.7). If the sequence g_ϵ converges to g in $w-L^1_{\text{loc}}(dt; w-L^1((1 + |v|^2)Mdvdx))$ then

$$(3.19) \quad \frac{q_\epsilon}{N_\epsilon} \rightarrow q = g' + g'_* - g - g_* \quad \text{in } w-L^1_{\text{loc}}(dt; w-L^1(d\mu dx)).$$

The propositions above apply only one fact about DiPerna-Lions solutions, the entropy inequality (2.40). For the full proof, see [16]. The fact that they are also weak solutions of the renormalized Boltzmann equation (3.5) leads to both the limiting dynamics, and subsequently a strong notion of convergence as in the following proposition.

Proposition 3.7 (Thanks to [16], the limiting Boltzmann equation theorem). *Let F be a given force field as in Theorem 2.2 such that*

$$F \in L^1_{\text{loc}}(dtdx; L^2(Mdv)).$$

Consider G_ϵ , a family of renormalized solutions of the Boltzmann initial-value problem (2.32) with initial data G_ϵ^{in} that satisfy the entropy bound (3.1). Let g_ϵ be the corresponding fluctuations (3.2). Then

- a) the sequence $(1 + |v|^2)g_\epsilon$ is relatively compact in $C([0, \infty); w-L^1(Mdvdx))$;*
- b) any convergent subsequence of g_ϵ has a limit $g \in C([0, \infty); L^2(Mdvdx))$ that is the unique solution of the initial-value problem*

$$(3.20) \quad \partial_t g + v \cdot \nabla_x g + F \cdot \nabla_v g + Lg = 0,$$

$$(3.21) \quad g(0) = g^{\text{in}} \equiv w-L^1\text{-}\lim_{\epsilon \rightarrow 0} g_\epsilon(0).$$

Proof ([16, 14]). Consider the renormalized Boltzmann equation (3.5) written in the form

$$(3.22) \quad \partial_t \gamma_\epsilon + (v \cdot \nabla_x + F \cdot \nabla_v) \gamma_\epsilon = \frac{1}{\epsilon} \frac{Q(G_\epsilon, G_\epsilon)}{N_\epsilon} = \int_{\mathbb{T}^3} \int_{\mathbb{S}^2} \frac{q_\epsilon}{N_\epsilon} b(v - v_*, \sigma) d\sigma M_* dv_*.$$

This means (see (2.37)), for every φ in $L^\infty(Mdv; C^1(\mathbb{T}^3))$ and every $0 \leq t_1 < t_2 < \infty$,

$$(3.23) \quad \int_{\mathbb{T}^3} \langle (\gamma_\epsilon(t_2) - \gamma_\epsilon(t_1)) \varphi \rangle dx - \int_{t_1}^{t_2} \int_{\mathbb{T}^3} \langle \gamma_\epsilon (v \cdot \nabla_x + F \cdot \nabla_v) \varphi \rangle dx dt = \int_{t_1}^{t_2} \int_{\mathbb{T}^3} \langle \frac{q_\epsilon}{N_\epsilon} \varphi \rangle dx dt.$$

Setting $z = \frac{1}{3}\epsilon g_\epsilon$ in the elementary inequality

$$(\log(1 + z))^2 \leq \frac{z^2}{1 + z} \quad \forall z > -1,$$

ones obtain

$$\gamma_\epsilon^2 \leq g_\epsilon^2 / N_\epsilon.$$

The nonlinear bound (3.17) in Proposition 3.5 then shows that the sequence γ_ϵ is bounded in $L^\infty(dt; L^2(Mdvdx))$ with

$$\int_{\mathbb{T}^3} \langle \gamma_\epsilon^2(t) \rangle dx \leq 2C^{\text{in}} \quad \forall t \geq 0.$$

Because the sequence γ_ϵ is bounded in $L^\infty(dt; L^2(Mdvdx))$ and $v \in L^2(Mdv)$, the sequence $\langle v \gamma_\epsilon \rangle$ is relatively compact in $w-L^1_{\text{loc}}(dt; w-L^2(dx))$. Since $F \in L^1_{\text{loc}}(dtdx; L^2(Mdv))$ as in the hypotheses, the sequence $\langle F \gamma_\epsilon \rangle$ is also relatively compact in $w-L^1_{\text{loc}}(dt; w-L^2(dx))$.

In (3.23), from right to left, the relative compactness of q_ϵ/N_ϵ , $|F|\gamma_\epsilon$ and $|v|\gamma_\epsilon$ in $w-L^1$ implies the map $t \mapsto \langle \gamma_\epsilon(t) \varphi \rangle$ is equicontinuous for each φ . Therefore, the sequence γ_ϵ , and thus g_ϵ is equicontinuous in $C([0, \infty); w-L^1(Mdvdx))$. The pointwise compactness of $g_\epsilon(t)$ is given by Proposition 3.1, hence the first assertion follows from the Arzelà-Ascoli theorem [20].

For the second assertion, ones pass to any convergent subsequence and take limits in (3.23) as $\epsilon \rightarrow 0$ to obtain

$$(3.24) \quad \int_{\mathbb{T}^3} \langle g(t_2) \varphi \rangle dx - \int_{\mathbb{T}^3} \langle g(t_1) \varphi \rangle dx - \int_{t_1}^{t_2} \int_{\mathbb{T}^3} \langle g(v \cdot \nabla_x + F \cdot \nabla_v) \varphi \rangle dx dt = \int_{t_1}^{t_2} \int_{\mathbb{T}^3} \langle \langle q \varphi \rangle \rangle dx dt,$$

where $q = g' + g'_* - g - g_*$ by Proposition 3.6. Substituting q into the formula of L in (2.43) shows that g is a weak solution of (3.20). But then g must therefore be the unique semigroup solution in $C([0, \infty); L^2(Mdvdx))$. \square

Now, it comes that with a careful choice of initial data G_ϵ^{in} , any L^2 solution of the linearized Boltzmann initial-value problem (2.42) can be attained to uniquely characterize the limiting fluctuation g . More specifically, notice that for every $g^{\text{in}} \in L^2(Mdvdx)$ ones can always construct a non-negative sequence $G_\epsilon^{\text{in}} = 1 + \epsilon g_\epsilon^{\text{in}}$ such that its fluctuations g_ϵ^{in} converge entropically to g^{in} , for example by choosing $g_\epsilon^{\text{in}} = \max\{g^{\text{in}}, -1/\epsilon\}$. With this preparation on the initial data G_ϵ^{in} , it holds in the following proposition that any approximating sequence of DiPerna-Lions fluctuations will even converge entropically for all positive time.

From [16] by Levermore for the case without external force, we maintain the main result about the linearized limit for the case with external force as follows.

Proposition 3.8 (Thanks to [16], the strong linearized limit theorem). *Let $b(z, \sigma)$ be a locally integrable collision kernel satisfying the DiPerna-Lions assumption (2.34). Let F be a given force field*

$$F(t, x, v) \in L_{\text{loc}}^1(dt dx; L^1(Mdv))$$

such that

$$(3.25) \quad \text{div}_v(F) = 0, \quad F \cdot v = 0 \quad \text{and} \quad F \in L_{\text{loc}}^1(dt dx; L^2(Mdv)) \cap L_{\text{loc}}^1(dt; W_{\text{loc}}^{1,1}(Mdvdx)).$$

Given any initial data $g^{\text{in}} \in L^2(Mdvdx)$, let

$$g \in C([0, \infty); L^2(Mdvdx))$$

be the unique solution of the initial-value problem

$$(3.26) \quad \partial_t g + v \cdot \nabla_x g + F \cdot \nabla_v g + Lg = 0, \quad g(0) = g^{\text{in}}.$$

For any sequence $G_\epsilon^{\text{in}} = 1 + \epsilon g_\epsilon^{\text{in}} \geq 0$ such that $g_\epsilon^{\text{in}} \rightarrow g^{\text{in}}$ entropically, let G^ϵ be any sequence of renormalized solutions of the Boltzmann initial value problem (2.32). Let g_ϵ and q_ϵ be the corresponding fluctuations (3.2) and scaled collision integrands (3.7). Then

- a) the sequence g_ϵ converges to g in $C([0, \infty); w\text{-}L^1(Mdvdx))$;*
- b) the sequence $g_\epsilon(t)$ converges entropically to $g(t)$ for every $t > 0$;*
- c) the sequence q_ϵ/N_ϵ converges to $g' + g'_\epsilon - g - g_\epsilon$ in $L_{\text{loc}}^1(dt; L^1((1 + |v|^2)d\mu dx))$.*

Proof. The proof is based on Proposition 3.7 (for assertion a), based on Proposition 3.1, the notion of entropic convergence, the squeezing argument, the dissipation equality (2.45), and

the entropy inequality part of (3.6) (for assertion b), and based on the last argument in the proof of Theorem 6.2 in [5] (for assertion c) [16]. \square

Remark 3.9. *The proof of Proposition 3.8 does not rely on the velocity averaging theory, which was used in establishing the basic global existence result of renormalized solutions [10] and in the related compactness investigation [17, 19] as well as in studying fluid dynamical limits [5, 15, 21, 4]. It is the regularity of the limiting linear dynamics as imposed in its dissipation **equality** together with the notion of entropic convergence that derives the strong convergence above.*

The notion of entropic convergence (thanks to Levermore) can be applied to any kinetic equation with a convex entropy. That includes kinetic equations with external forces as we are considering.

We wish to notice here that the result on linearization approximation (Proposition 3.8) can hold with more general external force (by reducing assumption on smoothness or exploring the very specific structure of the force as in the electromagnetic interaction within the plasmas [4] ...). In any case, the existence of renormalized solutions should be first verified or assumed, for the proof of Proposition 3.8, where the conditions on the force

$$\operatorname{div}_v F = 0, \quad F \cdot v = 0, \quad F \in L^1_{\text{loc}}(dtdx; L^2(Mdv))$$

are minimal requirements.

This paper handles short-range interactions only. It has not yet investigated, for kinetic equations with long-range interactions such as Coulombian interactions in plasmas, the linearization approximation and the hydrodynamical transition toward some models of fluid mechanics. On this trend, some questions arisen from the Vlasov-Maxwell-Boltzmann system have been nicely addressed by Arsénio and Saint-Raymond [4], and many problems are still open.

APPENDIX A. THE NOTION REGARDING SPACES

Throughout this paper, some of our notion regarding topological linear spaces is standard while some of it is less so. For a full description of the notion regarding spaces, we refer to [5], and many standard references therein.

Let E be any normed linear space; $\|\cdot\|_E$ denotes its norm and E^* denotes its dual space. Denote by $\langle \cdot, \cdot \rangle$ the natural bilinear form relating E^* and E . We will use the notion w - E to indicate the space E equipped with its weak topology, that is the coarsest topology on E for which each of the linear forms

$$u \mapsto \langle w, u \rangle_{E^*, E} \quad \text{for } w \in E^*,$$

is continuous.

Let X be a locally compact topological space and E a normed linear space. We will use the notion $C(X; w$ - $E)$ to indicate the space of continuous functions from X to w - E , that is the set of function u for which each of the linear forms

$$x \mapsto \langle w, u(x) \rangle_{E^*, E} \quad \text{is in } C(X) \quad \text{for each } w \in E^*.$$

We note that the Arzelà-Ascoli theorem holds on such spaces.

Let (Y, Σ, ν) be a measure space and E a normed linear space. For every $1 \leq p \leq \infty$, we will use the abbreviated notion $L^p(d\nu; E)$ for the Bochner space $L^p((Y, \Sigma, d\nu); E)$, containing all functions $u : Y \rightarrow E$ such that the corresponding norm is finite:

$$\|u\|_{L^p(d\nu; E)} := \left(\int_Y \|u(y)\|_E^p d\nu(y) \right)^{1/p} < +\infty \quad \forall \quad 1 \leq p < \infty,$$

$$\|u\|_{L^\infty(d\nu; E)} := \text{ess sup}_{y \in Y} \|u(y)\|_E = \min\{\alpha : \nu(\{y : \|u(y)\|_E > \alpha\}) = 0\} < +\infty.$$

We will use $L^p(d\nu)$ to denote the same space whenever E is a power of \mathbb{R} . For $1 \leq p < \infty$, the dual space of $L^p(d\nu; E)$ is $L^{p^*}(d\nu; E^*)$, where $p^* = p/(p-1)$. In this paper, only $p = 1, 2, \infty$ arise.

When Y is locally compact and $d\nu$ is a Borel measure, we will denote by $L_{\text{loc}}^p(d\nu; E)$ (or $L_{\text{loc}}^p(d\nu)$), the space determined by the family of seminorms

$$u \mapsto \left(\int_K \|u(y)\|_E^p d\nu(y) \right)^{1/p} \quad \text{for compact } K \subset Y.$$

For every $1 \leq p < \infty$, we will use the notation $w\text{-}L^p(d\nu; w\text{-}E)$ (or $w\text{-}L^p(d\nu)$) to denote the space $L^p(d\nu; E)$ equipped with its weak topology, that is the coarsest topology on $L^p((Y, \Sigma, d\nu); E)$ for which each of the linear forms

$$(A.1) \quad u \mapsto \int_Y \varphi(y) \langle w, u(y) \rangle_{E^*, E} d\nu(y) \quad \text{for } w \in E^* \quad \text{and } \varphi \in L^{p^*}(d\nu),$$

is continuous.

Finally, when Y is locally compact, we denote by $w\text{-}L_{\text{loc}}^p(d\nu; w\text{-}E)$ (or $w\text{-}L_{\text{loc}}^p(d\nu)$) the space $L_{\text{loc}}^p(d\nu; E)$ equipped with its weak topology; that is, the coarsest for which all the linear form (A.1) are continuous, where the functions u are restricted to have compact support in Y .

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